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NOTES ON DEGREE OF APPROXIMATION OF B-CONTINUOUS AND B-DIFFERENTIABLE FUNCTIONS

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Abstract

We give a quantitative variant of a recent Korovkin-type theorem for B-continuous functions, a refinement for B-differentiable functions, and applications of our results to estimate the degree of approximation of B-continuous and B-differentiable functions by certain linear operators. The applications include operators of Bernstein and Hermite-Fejer-type, also, a Jackson-type theorem for the space of B-continuous functions is given via the use of certain operators investigated by Szabados.

1. Introduction

The research in the present paper is a continuation of the recent article[1] in which a Korovkin-type theorem for the approximation of so-called Bögel-continuous (B-continuous) functions defined on the rectangle $R=[0,1]^2$ was proved. Here a real-valued function f on the (more general) rectangle $R=[a, b]\times[c, d]$ is called B-continuous if for every $(x, y)\in R$ there holds

$$\lim_{(u,v)\to(x,y)} \Delta_{u,v} f(x, y) = 0,$$

where $\Delta_{u,v} f(x, y) = f(x, y) - f(x, v) - f(u, y) + f(u, v)$.

The concepts of B-continuity (also denoted as $\Delta_{x,y}$-continuity) and of B-differentiabiliy (to be defined below) were introduced by Karl Bögel in 1934.

For more information on these notions the reader is referred to Bögel's papers[5]-[7]. It is easily verified that, under the pointwise operations of scalar multiplication and addition, the set $B(R)$ of Bögel-continuous functions constitutes a real vector space. Not very much appears to be known as far as further algebraic or topological properties of this space are concerned.

The continuation of our research on pointwise and uniform approximation of elements of the space $B(R)$ is mainly motivated by the fact that the space $B(R)$ is very closely related to what is nowadays denoted as "the approximation by Boolean sums of parametric extensions". The latter technipue is well known in Computer Aided Geometric Design and has recently also been investigated from the point of view of theoretical mathematics more thoroughly. See [10], [11], and the references cited there for further details and a partial survey Earlier (theoretical) work on the present subject was also carried out by I. Badea in the early 70's (see [1] for more references). Continuing his work, in [1] the following result was communicated.

Theorem A ([1, § 3]). Let $(L_{m,n})$, $(m, n) \in \mathbf{N}^2$, be a sequence of positive linear operators transforming functions of $B(R)$, $R=[0,1]^2$, into functions of $\mathbf{R}^R$ and satisfying

(i) $L_{m,n}(e;x,y) = 1$, where $e(s, t) = 1$.

For $f \in B(R)$ and $(x,y) \in R$, let

$$U_{m,n} f(x, y) = L_{m,n}[f(\cdot,y) + f(x,*) - f(\cdot,*); x, y]. \tag{1.1}$$

If the conditions

(ii) $L_{m,n}(\varphi;x,y) = x + U_{m,n}(x,y)$,

(iii) $L_{m,n}(\Psi;x,y) = y + V_{m,n}(x,y)$,

(iv) $L_{m,n}(\varphi^2+\Psi^2; x,y) = x^2+y^2+W_{m,n}(x,y)$,

are satisfied, where $\varphi(s,t) = s$, $\Psi(s,t) = t$, and $U_{m,n}(x,y)$, $V_{m,n}(x,y)$, and $W_{m,n}(x,y)$ converge to zero uniformly on R as m, n approach infinity. Then for every $f \in B(R)$ the sequence $(U_{m,n} f)$ converges uniformly to f on R.

In the present note we give several quantitative versions of Theorem A. The upper bounds will involve the so-called mixed modulus of continuity as introduced by A. Marchaud[15] and the total modulus of continuity, The first of these functionals is today also denoted as the $(1,1)$-modulus of continuity (see L.L. Schumaker [16, p.516]). As can be seen from its definition, namely

$$\omega_{mixed}(f; \delta_1, \delta_2) = \sup\{|\Delta_{x+h_1,y+h_2} f(x,y)|\},$$

where the supremum is taken over all $(x,y) \in R$, $(h_1,h_2) \in \mathbf{R}^2$, such that $(x+h_1, y+h_2) \in R$, $|h_1| \leqslant \delta_1$, $|h_2| \leqslant \delta_2$, and where $\Delta_{u,v} f(x,y)$ is defined as above, it is an appropriate quantity to be used when dealing with Bcontinuous functions. Indeed, in [2, Teorema 1] it was provet that the real-valued function f defined on R is uniformly B-continuous if and only if

$$\lim_{(\delta_1,\delta_2)\to(0,0)} \omega_{mixed}(f;\ \delta_1,\delta_2) = 0.$$

We recall that the function f which is real-valued and defined on R is uniformly B-continuous if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for every (x,y), $(x',y') \in R$ with $|x-x'| \leqslant \delta(\varepsilon)$ and $|y-y'| \leqslant \delta(\varepsilon)$ we have $|\Delta_{x',y'} f(x,y)| \leqslant \varepsilon$, and that B-continuous function on R is also uniformly B-continuous (see [7, Satz 7]).

For later purposes we note that the mixed modulus $\omega_{mixed}(f;\cdot,\delta_2)$ is a monotonically increasing function for each fixed δ_2; likewise $\omega_{mixed}(f;\delta_1,\cdot)$ is monotonically increasing for each fixed δ_1. Furthermore, for all non-negative numbers λ_1, λ_2 there holds

$$\omega_{mixed}(f;\ \lambda_1\cdot\delta_2,\ \lambda_2\cdot\delta_2) \leqslant (1+]\lambda_1[)\ (1+]\lambda_2[)\ \omega_{mixeb}(f;\delta_1,\delta_2), \quad (1.2)$$

where $]\lambda[$ denotes the largest integer smaller than λ.

§2 contains an estimate for $|U_{m,n}(f;\ x,\ y) - f(x,\ y)|$ where f is an arbitrary B-continuous function. In §3 we shall deal with the case where the function f is also Bögel-differentiable (B-differentiable), with a bounded B-derivative $D_B f$ (a definition is given below). Several applications are given in §4, including those for Bernstein operators, for the classical operators of Hermite-Fejèr, and for certain operators investigated by Szabados.

The methods employed in this paper are quite similar to those being frequently employed in the theory of approximation of univariate functions. However, B-continuity is a notion which differs remarkably in many respects from the usual continuity of bivariate functions (see Bögel [7]) so that special care has to be taken in the proofs.

2. Degree of Approximation of B-Continuous Functions

In this section we give estimates of the Mamedov-type for the operator $U_{m,n}$ defined as above. Mamedov[14] was the first to formulate theorems of the type which are nowadays known as Shisha-Mond-type theorems or as quantitative Korovkin theorems. Due to the lack of space we cannot discuss these historical matters any further here, but refer the reader to H. Gonska's thesis[10], for instance.

As far as the approximation of functions in B(R) is concerned, predecessors of the result in this section are I. Badea's dissertation[3] (especially

Capitolul III, §3) and his paper [2] where the special case was investigated in which the operator $L_{m,n}$ from above is the product of the parametric extensions of two univariate Bernstein operators. In this case, $U_{m,n}$ is the Boolean sum of the same parametic extensions (see [1] for the relevant definitions).

In order to arrive at a Mamedov-type inequality we first need an estimate for the difference $\Delta_{x,y} f$:

Lemma 2.1. For $f \in B(R)$, (x,y), $(s,t) \in R$, and δ_1, $\delta_2 > 0$ there holds

$$\max\{\Delta_{x,y} f(s,t),\ -\Delta_{x,y} f(s,t)\} = |\Delta_{x,y} f(s,t)| \tag{2.1}$$

$$\leqslant [1+]|s-x|/\delta_1[]\ [1+]|t-y|/\delta_2[]\ \omega_{mixed}(f;\delta_1,\delta_2) \tag{2.2}$$

$$\leqslant [1+|s-x|/\delta_1]\ [1+|t-y|/\delta_2]\ \omega_{mixed}(f;\delta_1,\delta_2).$$

Proof. The statement of the lemma follows from the definition of $\omega_{mixed}(f;\ \cdot,\ *)$ and (1.2).

Now we are in the state to prove a puantitative Korovkin-type theorem for B-continuous functions.

Theorem 2.2. Let $f \in B(R)$ and $U_{m,n} f$ be defined as in Theorem A. Then for every $\delta_1 > 0$, $\delta_2 > 0$ there holds:

$$\begin{aligned} |(f-U_{m,n} f)(x,y)| &\leqslant [1+\delta_1^{-1} L_{m,n}(|\cdot-x|;x,y)+\delta_2^{-1} L_{m,n}(|*-y|;x,y) \\ &\quad +(\delta_1\delta_2)^{-1} L_{m,n}(|\cdot-x|\cdot|*-y|;x,y)]\ \omega_{mixed}(f,\delta_1,\delta_2). \end{aligned} \tag{2.3}$$

Proof.

$$\begin{aligned} |(f-U_{m,n} f)(x,y)| &= |L_{m,n}(\Delta_{x,y} f(\cdot,*);x,y)| \\ &= \max\{L_{m,n}(\Delta_{x,y} f(\cdot,*);x,y), \\ &\qquad -L_{m,n}(\Delta_{x,y} f(\cdot,*);x,y)\} \\ &= \max\{L_{m,n}(\Delta_{x,y} f(\cdot,*);x,y), \\ &\qquad L_{m,n}(-\Delta_{x,y} f(\cdot,*);x,y)\}. \end{aligned} \tag{2.4}$$

The function on the right hand side of estimate (2.2) is continuous and thus B-continuous, i.e., we can apply $L_{m,n}$ to it. Because of the positivity of $L_{m,n}$ the above lemma yields

$$\begin{aligned} |(f-U_{m,n} f((x,y)| &\leqslant L_{m,n}[[1+|\cdot-x|/\delta_1]\ [1+|*-y|/\delta_2]\ \omega_{mixed}(f;\delta_1,\delta_2);x,y] \\ &= [1+\delta_1^{-1} L_{m,n}(|\cdot-x|;x,y)+\delta_2^{-1} L_{m,n}(|*-y|;x,y) \\ &\quad +(\delta_1\delta_2)^{-1} L_{m,n}(|\cdot-x||*-y|;x,y)]\ \omega_{mixed}(f;\delta_1,\delta_2), \end{aligned}$$

as stated.

Note that it may not be possible to apply $L_{m,n}$ directly to $|\Delta_{x,y} f|$ sinec this function might not be B-continuous (cf. Remark 2.4).

As is well known, it is quite cumbersome to handle the test functions figuring in Theorem 2.2 (namely $|\cdot - x|$, $|* - y|$, and $|\cdot - x|\,|* - y)$. In order to obtain an estimate involving only polynomials of coordinate degree 2, we apply the Cauchy-Schwarz inequality to the right hand side of (2.3) to arrive at

Corollary 2.3. Let $f \in B(R)$ and $U_{m,n} f$ be defined as in Theorem A. Then for every $\delta_1 > 0$, $\delta_2 > 0$ there holds

$$\begin{aligned} &|(f - U_{m,n} f)(x,y)| \\ &\leq [1 + \delta_1^{-1} [L_{m,n}((\cdot - x)^2; x,y)]^{1/2} + \delta_2^{-1} [L_{m,n}((* - y)^2; x,y)]^{1/2} \\ &+ (\delta_1 \delta_2)^{-1} [L_{m,n}((\cdot - x)^2 (* - y)^2; x,y)]^{1/2}]\, \omega_{mixed} (f, \delta_1, \delta_2). \end{aligned}$$

Remark 2.4. As announced following the proof of Theorem 2.2, here we give an example of a B-continuous function f on R such that the function $|\Delta_{s,t} f(\cdot, *)|$ is not B-continuous for every $s \neq 0$, $t \neq 0$: Let

$$f(x,y) = \begin{cases} h(x), & \text{for } x \neq 0,\ y = 0, \\ g(y), & \text{for } x = 0,\ y \neq 0, \\ h(0) + g(0), & \text{for } x = y = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where g and h are suitable univariate continuous functions. We have, for (x,y) lying in a sufficiently small ball around (x_0, y_0), that

$$|f(x_0, y_0) + f(x,y) - f(x_0, y) - f(x, y_0)| = 0, \quad \text{for } x_0 \neq 0,\ y_0 \neq 0,$$

$$|f(0, y_0) + f(x,y) - f(0,y) - f(x, y_0)| = |g(y_0) - g(y)|, \quad \text{for } y_0 \neq 0,$$

$$|f(x_0, 0) + f(x,y) - f(x_0, y) - f(x, 0)| = |h(x_0) - h(x)|, \quad \text{for } x_0 \neq 0,$$

$$\begin{aligned} |f(0,0) + f(x,y) - f(0,y) - f(x,0)| &= |h(0) + g(0) \\ - h(x) - g(y)| &\leq |h(0) - h(x)| + |g(0) - g(y)|. \end{aligned}$$

Hence the continuity of g and h implies that f is B-continuous. But, on the other hand, we have for $s \neq 0$, $t \neq 0$, $x \neq 0$, $y \neq 0$ that

$$\begin{aligned} &||\Delta_{s,t} f(0,0)| + |\Delta_{s,s} f(x,y)| - |\Delta_{s,t} f(0,y)| - |\Delta_{s,t} f(x,0)|| \\ &= ||h(0) - h(s) + g(0) - g(t)| - |h(x) - h(s)| - |g(y) - g(t)||. \end{aligned}$$

In general this does not tend to zero when x and y tend to zero, so $|\Delta_{s,t} f(\cdot, *)|$ is not B-continuous at $(0,0)$. Since we have $\Delta_{u,v}[\Delta_{s,t} f(\cdot, *)] = \Delta_{u,v} f(\cdot, *)$, the function $\Delta_{s,t}(\cdot, *)$ is B-continuous, so this example also furnishes an instance of a B-continuous function whose absolute value is not B-continuous.

3. Degree of Approximation of B-Differentiable Functions

In this section we consider the case of Bögel-Differentiable (B-differentiable) functions. we say that a function $f : R \to \mathbf{R}$ is B-differentiable if for every $(x,y) \in R$ there holds

$$\lim_{(u,v)\to(x,y)} \frac{\Delta_{u,v} f(x,y)}{(u-x)(v-y)} = D_B f(x,y) < \infty.$$

$D_B f$ is called the B-derivative of f.

We need the following mean value theorem for B-differentiable functions (cf. e. g. [7, Satz 17c]):

For every function f which is B-differentiable on R and for each (x,y), $(s,t) \in R$ there hofds

$$\Delta_{x,y} f(s,t) = (s-x)(t-y) D_B f(\xi,\eta) \tag{3.1}$$

for some suitable $\xi = \xi(s,t)$ between s and x, and $\eta = \eta(s,t)$ between t and y.

For functions f with a bounded B-derivative $D_B f$ we shall prove Theorem 3.1 below. Its second inequality is given in terms of the total modulus of continuity ω_{total} of the Bögel derivative $D_B f$. Here ω_{total} is given by

$$\omega_{total}(f;\ \delta_1, \delta_2) = \sup |f(x+h_1,\ y+h_2) - f(x,y)|,$$

where the supremum is taken over all $(x,y) \in R$, $(h_1, h_2) \in \mathbf{R}^2$ such that $(x+h_1,\ y+h_2) \in R$, $|h_1| \leqslant \delta_1$, $|h_2| \leqslant \delta_2$. $\omega_{total}(f;\ \cdot, *)$ is monotonically increasing with respect to both variables, i.e. $\gamma_i \leqslant \delta_i$ for $i=1,\ 2$ implies

$$\omega_{total}(f;\ \gamma_1,\ \gamma_2) \leqslant \omega_{total}(f;\ \delta_1,\ \delta_2).$$

Another property is that, for all non-negative numbers λ_1, λ_2, we have

$$\omega_{total}(f;\ \lambda_1 \cdot \delta_1,\ \lambda_2 \cdot \delta_2) \leqslant [1 + \max\{]\lambda_1[,\ ]\lambda_2[\}]\ \omega_{total}(f;\ \delta_1,\ \delta_2).$$

Also, the total modulus ω_{total} is related the so-called partial moduli of continuity $\omega_{total}(f;\ \delta_1,\ 0)$ and $\omega_{total}(f;\ 0,\ \delta_2)$ by the incpuality

$$\omega_{total}(f;\ \delta_1,\ \delta_2) \leqslant \omega_{total}(f;\ \delta_1,\ 0) + \omega_{total}(f;\ 0,\ \delta_2).$$

See, e. g., A. F. Timan[20] for some further information on these measures of smoothness.

Theorem 3.1. Let again $L_{m,n}$ and $U_{m,n}$ be as in Theorem A. If f has a bounded B-derivative $D_B f$, the following inequalities hold:

$$\text{(i)}\quad |(f - U_{m,n} f)(x,y)| \leqslant \|B_B f\|_\infty L_{m,n}[|\cdot - x|\ |* - y|;\ x,y], \tag{3.2}$$

$$\text{(ii)}\quad |(f - U_{m,n} f)(x,y)| \leqslant |D_B f(x,y)|\ |L_{m,n}[(\cdot - x)(* - y);\ x,y]| + [L_{m,n}[|\cdot - x|\ |* - y|;\ x,y] + \delta_1^{-1} L_{m,n}[(\cdot - x)^2 |* - y|;x,y] + \delta_2^{-1} L_{m,n}[|\cdot - x|(* - y)^2;\ x,y]]\ \omega_{total}(D_B f;\ \delta_1, \delta_2)$$

for each δ_1, $\delta_2 > 0$.

Proof. From the mean value theorem for B-differentiable functions we know that there exist a $\xi = \xi\,(s,t)$ between s and x as well as an $\eta = \eta\,(s,t)$ between t and y such that

$$\Delta_{x,y} f\,(s,t) = D_B f\,(\xi,\eta)\,(x-s)\,(y-t).$$

Hence it follows that

$$\max\,[\Delta_{x,y} f\,(s,t),\ -\Delta_{x,y} f\,(s,t)\,] \leqslant \|D_B f\|_\infty\,|x-s|\,|y-t|.$$

Thus

$$\begin{aligned} |\,(f-U_{m,n} f)\,(x,y)\,| &= |\,L_{m,n}\,[\Delta_{x,y} f\,(\cdot,*)\,;\ x,y]\,| \\ &\leqslant \|D_B f\|_\infty\, L_{m,n}\,[\,|\cdot-x|\,|*-y|;\ x,y], \end{aligned}$$

proving (i) . Furhermore,

$$\begin{aligned} \Delta_{x,y} f\,(s,\ t) = {} & [D_B f\,(\xi,\ \eta) - D_B f\,(x,\ y)\,]\,(s-x)\quad(t-y) \\ & + D_B f\,(x,\ y)\quad(s-x)\quad(t-y). \end{aligned}$$

Since the left hand side, as well as the second summand of the right hand side are B-continuous, we know that the first term of the right hand side is also B-continuous. Hence we can write

$$\begin{aligned} & |\,L_{m,n}\,[\Delta_{x,y} f\,(\cdot,*)\,;\ x,y]\,| \\ = {} & |\,L_{m,n}\,[\,\{D_B f\,(\xi\,(\cdot,*)\,,\ \eta\,(\cdot,*))\ - D_B f\,(x,y)\,\}\,(\cdot-x)\,(*-y);x,y] \\ & + L_{m,n}\,[D_B f\,(x,\ y)(\cdot-x)\,(*-y);\ x,\ y]\,| \\ \leqslant {} & |\,L_{m,n}\,[\{D_B f\,(\xi\,(\cdot,*),\ \eta\,(\cdot,*))\ - D_B f\,(x,y)\,\}\,(\cdot-x\,((*-y)\,;x,y]\,| \\ & + |D_B f\,(x,y)\,|\,|\,L_{m,n}\,[\,(\cdot-x)\,(*-y)\,;\ x,y]\,| \\ = {} & \max\,\{L_{m,n}\,(g;x,y)\ L_{m,n}\,(-g;x,y)\,\} \\ & + |D_B f\,(x,y)\,|\,|\,L_{m,n}\,[\,(\cdot-x)\quad(*-y)\,;\ x,\ y]\,| \end{aligned} \tag{3.3}$$

with $g\,(s,t) = \{D_B f\,(\xi,\eta)\ - D_B f\,(x,y)\,\}\,(s-x)\quad(t-y)$. Now

$$\begin{aligned} \max\,\{g\,(s,t)\,,\ -g\,(s,t)\,\} &= |D_B f\,(\xi,\eta)\ - D_B f\,(x,y)\,|\,|s-x|\,|t-y| \\ &\leqslant \omega_{total}\,[D_B f;\ |s-x|,\ |t-y|]\,|s-x|\,|t-y| \\ &\leqslant [1+\max\,\{|s-x|\,\delta_1^{-1},\ |t-y|\,\delta_2^{-1}\}]\,|s-x|\,|t-y|\,\omega_{total}(D_B f;\ \delta_1,\delta_2) \\ &\leqslant [\,|s-x|\,|t-y| + \delta_1^{-1}\,(s-x)^2\,|t-y| + \delta_2^{-1}\,|s-x|\,(t-y)^2]\,\cdot \\ &\qquad \omega_{total}\,(D_B f;\ \delta_1,\ \delta_2). \end{aligned}$$

This, together with (3.3) , yields the claim of (ii) .

4. Applications

Here we shall give applications of our general theorems to those operatos $U_{m,n}$ which are defined on the basis of the well-known univariate Bernstein and Hermite-Fejèr operators, and on the basis of some interesting positive operators considered a few years ago by Szabados. All three operators are discretely defined so that the resulting operators $U_{m,n}$ are defined for function in B(R) with R being a suitable rectangle. We remark that the observations of this section are quite analogous to various recent results obtained by one of the authors in [10], and dealing with the approximation of functions $C([-1,1]^2)$ and in $C^{1,1}([-1,1]^2)$. Here the latter symdols benote the spaces of all continuous functions on $[-1,1]^2$, and of all functions having continuous partials up to order $(1,1)$, respectively.

4.1. *Bernstein-Type Operators.* The univariate Bernstein operator (see e. g. [13])

$$B_m f(x) = \sum_{k=0}^{m} f\left[\frac{k}{m}\right]\left[\begin{matrix} m \\ k \end{matrix}\right] x^k (1-x)^{m-k},\ f \in R^{[0,1]},\ x \in [0,1],$$

reproduces linear functions. Furthermore, it is known that

$$B_m(|\cdot - x|;\ x) \leqslant [B_m((\cdot - x)^2;\ x)]^{1/2} = [x(1-x)/m]^{1/2}. \qquad (4.1)$$

Hence Theorem 2.2 yields for any $f \in B(R)$ and the operators $U_{m,n}$ defined via $L_{m,n} = {}_xB_m \circ {}_yB_n$ the inequality

$$\begin{aligned} &|(f - U_{m,n} f)(x,y)| \\ &\leqslant [1 + \delta_1^{-1} [x(1-x)/m]^{1/2} + \delta_2^{-1} [y(1-y)/n]^{1/2} \\ &+ \delta_1^{-1}\delta_2^{-1} [x(1-x)/m]^{1/2} [y(1-y)/n]^{1/2}]\ \omega_{mixed}(f;\ \delta_1, \delta_2) \\ &= [1 + \delta_1^{-1} [x(1-x)/m]^{1/2}][1 + \delta_2^{-1} [y(1-y)/n]^{1/2}] \cdot \\ &\omega_{mixed}(f;\ \delta_1, \delta_2). \end{aligned}$$

Choosing $\delta_1 = 1/\sqrt{m}$, $\delta_2\sqrt{n}$ we arrive at

$$\begin{aligned} &|(f - U_{m,n} f)(x,y)| \\ &\leqslant [1 + [x(1-x)]^{1/2}][1 + [y(1-y)]^{1/2}]\ \omega_{mixed}(f; 1/\sqrt{m}, 1/\sqrt{n}) \\ &\leqslant \frac{9}{4}\ \omega_{mixed}(f;\ 1/\sqrt{m},\ 1/\sqrt{n}). \end{aligned}$$

This inequality was proved in a paper by one of the presnent authors[2] However, due to the fact that the Bernstein operators are discretely defined, we can also proceed as follows is order to obtain a better constant. To this end we recall inequality (2.1) and observe that, for the special case considered now, the estimate (2.3) can be replaced by

$$\begin{aligned} &|(f - U_{m,n} f)(x,y)| \\ &\leqslant ({}_xB_m \circ {}_yB_n)\ [\ [1 +]|\cdot - x|/\delta_1[\,]\ [1 +]|* - y|/\delta_2[\,]\ ;x,y] \cdot \\ &\omega_{mixed}(f;\ \delta_1,\ \delta_2) \\ &= B_m\ [1 +]|\cdot - x|/\delta_1[;\ x]\ B_n\ [1 +]|* - y|/\delta_2[;\ y] \cdot \\ &\omega_{mixed}(f;\ \delta_1,\ \delta_2). \end{aligned}$$

Picking again $\delta_1 = 1/\sqrt{m}$, $\delta_2 = 1/\sqrt{n}$ now gives the upper bound

$$\begin{aligned} &[1 + B_m\ [\,]|\cdot - x|\sqrt{m}[;\ x]]\ [1 + B_n\ [\,]|* - y|\sqrt{n}[;\ y]] \cdot \\ &\omega_{mixed}(f;\ 1/\sqrt{m},\ 1/\sqrt{n}). \end{aligned}$$

The terms involving B_m and B_n were evaluated in P. C. Sikkema's papers[17],[18] (any reader interested in the very detalls is referred to E. Blaswich's tnesis [4]). As shown in Sikkema's second paper it is true that for all $k \in \mathbf{N}$ there holds

$$B_k[|\cdot - z|\sqrt{k}; z] \leq \left[\frac{4306+837\sqrt{6}}{5832} - 1\right] = c_1 - 1 \approx 0.089887,$$

$$0 \leq z \leq 1.$$

Together with the above inequality this implies

$$|(f - U_{m,n}f)(x,y)| \leq c_1^2\, \omega_{mixed}(f;\ 1/\sqrt{m},\ 1/\sqrt{n}).$$

This special consequence of our general theorem on the approximation of B-continuous function was also obtained in the dissertation of one of the present authors ([3, Teorema 3.1.1]) but as a corollary of a different generalization. The above estimate is very similar to that of Sikkema[18] who proved that

$$|(f - B_k f)(z)| \leq c_1\, \omega(f,\ 1/\sqrt{k}),$$

where $\omega(f, \cdot)$ is the usual univariate modulus of continuity. Moreover, c_1 is the best constant. Continuing this analogy, we may prove that c_1^2 is the best constant in the above estimate for $|(f - U_{m,n}f((x,y)|$. Indeed, if the real-valued functions f_1 and f_2 defined on $[0, 1]$ satisfy

$$\sup\{|(f_1 - B_m f_1)(x)| : x \in [0,1]\} = c_1\, \omega(f_1;\ 1/\sqrt{m})$$

and

$$\sup\{|(f_2 - B_n f_2(y)| : y \in [0,1]\} = c_1\, \omega(f_2;\ 1/\sqrt{n}),$$

(the existence of these two functions is guarannteed by Sikkema's result), then for the product function f_0 given by

$$f_0(x,y) = f_1(x)\, f_2(y),\quad x,\ y \in [0,1],$$

we have

$$\sup_{(x,y)\in[0,1]^2} |(f_0 - U_{m,n}f_0)(x,y)|$$
$$= \sup_{x\in[0,1]} |(f_1 - B_m f_1((x)| \sup_{y\in[0,1]} |(_2 - B_n f_2)(y)|.$$

Hence $\omega_{mixed}(f_0;\ \delta_1, \delta_1) = \omega(f_1;\ \delta_1)\,\omega(f_2, \delta_2)$ implies

$$\sup_{(x,y)\in[0,1]^2} |(f_0 - U_{m,n}f_0)(x,y)| = c_1^2\, \omega_{mixed}(f;\ 1/\sqrt{m},\ 1/\sqrt{n}).$$

This equality, together with the general estimate

$$|(f - U_{m,n}f)(x,y)| \leq c_1^2\, \omega_{mixed}[f;\ 1/\sqrt{m},\ 1/\sqrt{n}),$$

shows that c_1^2 is the best constant in the above Popoviciu-type estimate.

Consider next the case where f is a function defined on $[0,1]^2$, and possessing a bounded Bögel derivative $D_B f$ everywhers in that interval. In this case statement (i) of the above Theorem 3.1 implies

$$\begin{aligned} &|(f-U_{m,n}f)(x,y)| \\ &\le \|D_B f\|_\infty L_{m,n}(|\cdot-x||*-y|;\ x,y) \\ &= \|D_B f\|_\infty B_m(|\cdot-x|;\ x)\, B_n(|*-y|;\ y) \\ &\le \|D_B f\|_\infty [x(1-x(/m]^{1/2}[y(1-y(/n]^{1/2} \\ &= O(1/\sqrt{m})\ O(1/\sqrt{n}) \end{aligned}$$

Using inequality (ii) in Theorem 3.1 and observing that $L_{m,n}[(\cdot-x)\cdot(*-y);\ x,y] = 0$, it follows that

$$\begin{aligned} &|(f-U_{m,n}f)(x,y)| \\ &\le [L_{m,n}[|\cdot-x||*-y|;\ x,y] + \delta_1^{-1}L_{m,n}[(\cdot-x)^2|*-y|;\ x,y] \\ &\quad + \delta_2^{-1}L_{m,n}[|\cdot-x|(*-y)^2;\ x,y]]\,\omega_{total}(D_B f;\ \delta_1,\delta_2) \\ &\le [[x(1-x)/m]^{1/2}[y(1-y)/n]^{1/2} \\ &\quad + \delta_1^{-1}x(1-x)/m\,[y(1-y)/n]^{1/2} \\ &\quad + \delta_2^{-1}[x(1-x)/m]^{1/2}\,y(1-y)/n]\ \omega_{total}(D_B f;\ \delta_1,\delta_2). \end{aligned}$$

For $(x,y) \in (0,1)^2$ we pick $\delta_1 = [x(1-x)/m]^{1/2}$, $\delta_2 = [y(1-y)/n]^{1/2}$, yielding

$$\begin{aligned} &|(f-U_{m,n}f)(x,y)| \le 3\,[x(1-x)/m]^{1/2}[y(1-y)/n]^{1/2} \\ &\quad \cdot\, \omega_{total}[D_B f;\ [x(1-x)/m]^{1/2},\ [y(1-y)/n]^{1/2}]. \end{aligned}$$

The above estimate is also true if x or y are equal to 0 or 1. Because $D_B f$ is bounded, this estimate implies the same order of approximation as given above, namely $O(1/\sqrt{m})\ O(1/\sqrt{n})$. However, the occurence of $\omega_{total}(D_B f;\ \cdot,*)$ enables us to take advantage of smoothness properties of $D_B f$. See the following section for a related discussion with respect to sequences of certain interpolation operators.

4.2 *Heremite-Fejer-type Operators.* In order to give an application of somewhat different flavour we consider a special case of the classical Hermite-Fejer operators H_m as introduced by L. Fejer[8] in 1916. In the sequel $H_m f$ will denote the polynomial of degree $2m-1$ interpolating a function $f \in R^{[-1,1]}$ at the roots of the Cebysev polynomial T_m of the first kind, i. e. $T_m(x) = \cos(m \cdot \arccos x)$, and having derivatives equal to zero at these points. For $m \ge 2$ and $|x| \le 1$ the following (in)equalities hold (see [9], [10] for details):

$$H_m(\cdot - x;\ x) = -m^{-1}\,T_m(x)\,T_{m-1}(x), \tag{4.2}$$

$$H_m(|\cdot - x|;\ x) \le 4m^{-1}\,|T_m(x)|\,[(1-x^2)^{1/2}\,ln\,m + 1], \tag{4.3}$$

$$H_m((\cdot - x)^2;\ x) = m^{-1}\,T_m^2(x). \tag{4.4}$$

Defining $U_{m,n}$ on the basis of $L_{m,n} = {}_xH_m \circ {}_yH_n$ for $m, n \ge 2$, an application of Theorem 2.2 and of (4.2) − (4.4) yields

$$|(f-U_{m,n}f)(x,y)|$$

$$\leq [1+\delta_1^{-1}\, 4m^{-1}\,|T_m(x)|\,[(1-x^2)^{1/2}\ln m+1]$$
$$+\delta_2^{-1}\, 4n^{-1}\,|T_n(y)|\,[(1-y^2)^{1/2}\ln n+1]$$
$$+\delta_1^{-1}\delta_2^{-1}\, 16m^{-1}n^{-1}\,|T_m(x)\,T_n(y)|\cdot$$
$$[(1-x^2)^{1/2}\ln m+1][(1-y^2)^{1/2}\ln n+1]]\,\omega_{mixed}(f;\delta_1,\delta_2)$$

for all $f\in B(R)$ and all $(x,y)\in[-1,1]^2$. If x is not a zero of T_m, and if y is none of T_n, we may pick

$$\delta_1=m^{-1}\,|T_m(x)|\,[(1-x^2)^{1/2}\ln m+1],$$
$$\delta_2=n^{-1}\,|T_n(y)|\,[(1-y^2)^{1/2}\ln n+1],$$

to obtain

$$|(f-U_{m,n}f)(x,y)|$$
$$\leq 25\omega_{mixed}[f;\ m^{-1}\,|T_m(x)|\,[(1-x^2)^{1/2}\ln m+1],$$
$$n^{-1}\,|T_n(y)|\,[(1-y^2)^{1/2}\ln n+1]].$$

Because of the interpolation properties of the Hermite-Fejér operators, this estimate also holds if x is a zero of T_m, or if y is a zero of T_n. Thus we have tne uniform estimate

$$|(f-U_{m,n})(x,y)|\leq 25\omega_{mixed}[f;m^{-1}(\ln m+1),\ n^{-1}(\ln n+1)].$$

Estimates analogous to those for the above Boolean sum of the parametric extensions of two univariate Bernstein are also available in the case in which the function f has a bounded B-derivative D_Bf. Here, the quantity

$$25\,\omega_{mixed}[f;\ m^{-1}(\ln m+1),\ n^{-1}(\ln n+1)]$$

s bounded from above by

$$25\|D_Bf\|_\infty\, m^{-1}(\ln m+1)\, n^{-1}(\ln n+1)=0\,(m^{-1}\ln m)\cdot 0\,(n^{-1}\ln n).\tag{4.5}$$

The same order of approximation can be obtained if we apply Theorem 3.1 (i) to the above situation, but with the better constant 16 (rather tnan 25). An analogous improvement can always be achieved if the operator $L_{m,n}$ is the product of the parametric extensions of two univariate positive operators L_m and L_n. This is due to the fact that the direct approach towards $\|D_Bf\|_\infty$ as used in the proof of Theorem 3.1 (i) avoids certain intermediate steps which were necessary to get the more general inequality of Theorem 2.2.

Now we show that the order of approximation becomes better once we assume certain regularity properties of D_Bf. In this case, inequality (ii) of Theorem 3.1 first gives

$$|(f-U_{m,n}f)(x,y)|$$
$$\leq\|D_Bf\|_\infty\,|H_m(\cdot-x;x)|\,|H_n(*-y;y)|$$
$$+[H_m(|\cdot-x|;x)\,H_n(|*-y|;y)\,\delta_1^{-1}H_m((\cdot-x)^2H_n(|*-y|;y)$$
$$+\delta_2^{-1}H_m(|\cdot-x|;x)\,H_n((*-y)^2;y)]\,\omega_{total}(D_Bf;\delta_1,\delta_2)$$

$$\leq \|D_B f\|_\infty m^{-1} n^{-1}$$
$$+[4m^{-1}(lnm+1)4n^{-1}(lnn+1)+\delta_1^{-1}m^{-1}4m^{-1}(lnn+1)$$
$$+\delta_2^{-1}4m^{-1}(lnm+1)n^{-1}]\omega_{total}(D_B f;\delta_1,\delta_2).$$

Chossing $\delta_1=(lnm+1)^{-1}$ and $\delta_2=(lnn+1)^{-1}$ yields

$$|(f-U_{m,n}f)(x,y)|\leq \|D_B f\|_\infty m^{-1}n^{-1}$$
$$+[16m^{-1}(lnm+1)n^{-1}(lnn+1)+m^{-1}(lnm+1)4n^{-1}(lnn+1)$$
$$+4m^{-1}(lnm+1)n^{-1}(lnn+1)]\omega_{total}[D_B f;(lnm+1)^{-1},$$
$$(lnn+1)^{-1}].$$

Assume, for example, that $D_B f$ has bounded first order partials $D^{(1,0)}(D_B f)$ and $D^{(0,1)}(D_B f)$. Then the relationship between ω_{total} and the partial moduli of continuity mentioned above implies

$$\omega_{total}[D_B f;\ (lnm+1)^{-1},\ (lnn+1)^{-1}]$$
$$\leq \|D^{(1,0)}(D_B f)\|_\infty (lnm+1)^{-1}+\|D^{(0,1)}(D_B f)\|_\infty (lnn+1)^{-1}$$
$$\leq 2\max\{\|D^{(1,0)}(D_B f)\|_\infty,\ \|D^{(0,1)}(D_B f)\|_\infty\}[(lnm+)^{-1}+(lnn+1)^{-1}].$$

Hence with $c_2=48\max\{\|D^{(1,0)}(D_B f)\|_\infty,\ \|D^{(0,1)}(D_B f)\|_\infty\}$,

$$\|f-U_{m,n}\|_\infty$$
$$\leq \|D_B f\|_\infty m^{-1}n^{-1}+c_2 m^{-1}(lnm+1)n^{-1}(lnn+1)\cdot[(lnm+1)^{-1}+(lnn+1)^{-1}]$$
$$=\|D_B f\|_\infty m^{-1}n^{-1}+c_2[m^{-1}n^{-1}(lnn+1)+m^{-1}(lnm+1)^{-1}].$$

Letting $m=n$ for the sake of simplicity, the latter expression is of order $0[n^{-2}lnn]=0[(n^{-1}lnn)^2]$, $n\to\infty$, and thus gives a better degree of approximation than that implied by (4.5), or by statement (i) of Therom 3.1.

4.3 *A Jackson-type Theorem for B-Continuous Functions.* § 4.1 and 4.2 implicitly contain short proofs of the Weierstrass approximation theorem for B-continuous functions and pseudopolynmials, i.e., of the fact that every B-continuous function on $[0,1]^2$ can be uniformly approximated by a sequence of functions of the type

$$P_{m,n}(x,y)=\sum_{k=0}^{m}a_k(y)x^k+\sum_{k=0}^{n}b_k(x)y^k, \tag{4.6}$$

where a_k and b_k are arbitrary univariate functions on $[0,1]$. §4.1, for example, contains the quantitatve assertion that

$$E_{m,n}f\leq c_3\ \omega_{mixed}(f;\ 1/\sqrt{m},\ 1/\sqrt{n}), \tag{4.7}$$

where $E_{m,n}f=\inf\|f-p_{m,n}\|_\infty$, the infimum being taken over all functions of the type (4.6), and c_3 denoting a positive constant independent of f,m, and n. For the case of continuous functions and continuous coefficient functions in (4.6) it is known (cf. [12]) that the order of approximation in (4.7) may be improved. More exactly, there holds the Jackson-type estimate

$$E_{m,n}f\leq c_4\ \omega_{mixed}(f;\ m^{-1},\ n^{-1}). \tag{4.8}$$

This inequality is optimal in the sense that the approximation order cannot be improved. Similarly as in §4.1, this can be verified using functions of the product type and the corresponding univariate results. Theorem 2.2 enables us to show that (4.8) also holds for B-continuous functions.

In [19] J. Szabados constructed certain positive linear and discretely defined operators $L_n : N(-1/4, 1/4) \to \Pi_n$ which give the Jackson order of approximation, that is

$$\|f - L_n f\|_\infty \leqslant c_5^* \omega (f; n^{-1}), \text{ for } f \in C[-1/4, 1/4].$$

Substituting the function $f_x(t) = |t - x|$ into this estimate we have

$$L_n (|\cdot - x|; x) \leqslant \|L_n f_x - f_x\|_\infty \leqslant c_5^*/n.$$

Therefore, according to Theorem 2.2, and with $\delta_1 = m^{-1}$, $\delta_2 = n^{-1}$, we have for the corresponding operators $U_{m,n}$ that

$$|(f - U_{m,n} f)(x,y)| \leqslant (1 + c_5^*)^2 \omega_{mixed} (f; m^{-1}, n^{-1})$$

for every function $f \in C[[-1/4, 1/4]^2]$. Since $U_{m,n}$ yields approximants of the form (4.6), this means that the estimate (4.8) is valid for all such functions. Obviously an adequate transformation also gives (4.8) for functions $f \in B(R)$, where R is an arbitrary rectangle.

For the operators of J. Szabados we do not have a useful representation (estimate) for $L_n(\cdot - x; x)$ which would allow us to give meaningful applications of the results of §3 for B-differentiable functions as well.

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