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SIMULTANEOUS APPROXIMATION AND GLOBAL SMOOTHNESS PRESERVATION

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1. Introduction

Over the recent years there has been considerable interest in the preservation of global smoothness properties by linear operators. Investigations of this kind might have their historical roots in the famous 1951 paper of Stečkin [27] who showed, for example, the following

Theorem A.
For fixed $s, n \in \mathbb{N}$ and $f \in C_{2\pi}$, let t_n be a trigonometric polynomial of degree $\leq n$ such that

$$\| f - t_n \| \leq C_1 \cdot \omega_s(f ; n^{-1}) .$$

Then for all $\delta > 0$ one has

$$\omega_s(t_n;\delta) \leq (\sin\tfrac{1}{2})^{-s} \cdot (1+2^s \cdot C_1) \cdot \omega_s(f;\delta).$$

Here ω_s is the well-known modulus of smoothness of order s as introduced, e.g., in Schumaker's book [24].

A result on smoothness preservation by the well-known Bernstein operators B_n on $C[0,1]$ was given in 1965 by Hajek [14]:

Theorem B.
Let $f \in Lip_M(1;[0,1])$. Then $B_n f \in Lip_M(1;[0,1])$.

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This result was later generalized by Lindvall [19] and Brown/Elliott/Paget [4] who showed that the same statement is also true if $\mathrm{Lip}_M(1;[0,1])$ is replaced by $\mathrm{Lip}_M(\alpha;[0,1])$, $0 < \alpha \leq 1$. This means that, if global smoothness of a function $f \in C[0,1]$ is expressed by stating that it satisfies a certain Lipschitz condition, then the same is true for its approximant $B_n f$.

Theorem B was generalized in 1991 by Anastassiou and the present authors in [1]. We recall

Theorem C.
For the Bernstein operators B_n one has, for all $f \in C[0,1]$ and $\delta \geq 0$,

$$\omega_1(B_n f;\delta) \leq 1 \cdot \tilde{\omega}_1(f;\delta) \leq 2 \cdot \omega_1(f;\delta) .$$

Here $\tilde{\omega}_1(f\,;\cdot)$ denotes the least concave majorant of $\omega_1(f\,;\cdot)$. The constants 1 and 2 are best possible.

Note that the first inequality of Theorem C implies the result of Lindvall and Brown/Elliott/Paget mentioned before, but the second does not.

Theorem C can in turn be generalized by a statement in which global smoothness preservation is expressed in terms of Peetre K-functionals of order s, $s \geq 1$, given by

$$K_s(f;\delta) := K\big(f;\delta;C[0,1],C^s[0,1]\big) = \inf\{\|f-g\| + \delta\cdot\|g^{(s)}\| : g \in C^s[0,1]\} .$$

Here, $f \in C[0,1]$ and $\delta \geq 0$. For the case of Bernstein operators this statement reads as follows.

Theorem D.
For the operators B_n one has for all $s \in \mathbb{N}$, $f \in C[0,1]$ and $\delta \geq 0$ the inequalities

$$K_s(B_n f;\delta) \leq 1\cdot K_s\Big(f\,,\frac{(n)_s}{n^s}\cdot\delta\Big) \leq K_s(f;\delta).$$

Here, $(n)_s = \prod_{\beta=0}^{s-1} (n-\beta)$.

For the case $s = 1$, Theorem D in combination with Brudnyi's representation theorem (cf. [21]) implies Theorem C. The inequalities given in Theorem D follow from a more general assertion which we will prove below.

The preservation of global smoothness properties was also considered in other contexts. As three examples we mention here certain multivariate cases (see [2], [5]), the approximation of stochastic processes (see [3]) and the preservation of global smoothness by Bézier curves (see [23],[15]). Also the research in [7], [9], [10], [16], [17] and [22] is related to various aspects of global smoothness preservation.

The research in our present note is motivated by certain approximation properties of the classical Bernstein operators. As is well known (from the classical paper of Wigert [28], for example), for any $f \in C^k[0,1]$ the sequence of k-th derivatives of its Bernstein polynomials converges to the k-th derivative $D^k f = f^{(k)}$ uniformly on $[0,1]$. It is thus natural to investigate the problem whether simultaneous approximation processes in the above sense also preserve global smoothness of the derivatives of a k-times differentiable function f. Several results of this type for both the univariate and the multivariate cases will be presented in this paper.

With respect to notation in this article we mention that for a function f continuous on a compact set K, $\|f\|$ will always denote its sup norm. If a different norm is needed this will be explicitly indicated.

2. General Estimates in terms of K-Functionals

We first show a generalized version of Theorem D for a certain class of operators which includes those of Bernstein as special cases.

Theorem 2.1.
Let $k \geq 0$ and $s \geq 1$ be integers, and let $I := [a,b]$ and $I' := [c,d] \subseteq [a,b]$ be compact intervals with non-empty interior. If L is a linear operator, satisfying

$$L: C^k(I) \to C^k(I') \text{ such that } \|D^k Lf\|_{I'} \leq a_{k,L} \cdot \|f^{(k)}\|_I \,,\; a_{k,L} \neq 0,$$

for all $f \in C^k(I)$, as well as

$$L: C^{k+s}(I) \to C^{k+s}(I') \text{ with } \|D^{k+s} Lg\|_{I'} \leq b_{k,s,L} \cdot \|g^{(k+s)}\|_I$$

for all $g \in C^{k+s}(I)$, then for all $f \in C^k(I)$ and all $\delta \geq 0$ one has

$$K_s(D^k Lf\,;\delta)_{I'} \leq a_{k,L} \cdot K_s\left(f^{(k)}\,; \frac{b_{k,s,L}}{a_{k,L}} \cdot \delta \right)_I .$$

Proof.
Let $f \in C^k[a,b]$ and $g \in C^{k+s}[a,b]$ be arbitrarily given. Then the sublinearity of K_s gives

$$K_s(D^k Lf\,;\delta)_{I'}$$

$$= K_s(D^k L(f - g + g) ; \delta)_{I'}$$

$$\leq K_s(D^k L(f - g) ; \delta)_{I'} + K_s(D^k Lg ; \delta)_{I'}$$

$$\leq \|D^k L(f - g)\|_{I'} + \inf\{\|D^k Lg - h\|_{I'} + \delta \cdot \|D^s h\|_{I'} : h \in C^s[c,d]\}$$

Taking $h = D^k Lg \in C^s[c,d]$ in the second term shows that

$$K_s(D^k Lf ; \delta)$$

$$\leq \|D^k L(f - g)\|_{I'} + \delta \cdot \|D^{k+s} Lg\|_{I'}$$

$$\leq a_{k,L} \cdot \|D^k(f - g)\|_I + \delta \cdot b_{k,s,L} \cdot \|D^{k+s} g\|_I$$

$$= a_{k,L} \cdot \left\{\|D^k (f - g)\|_I + \frac{b_{k,s,L}}{a_{k,L}} \cdot \delta \cdot \|D^{k+s} g\|_I\right\}.$$

Passing to the infimum over $h = D^k g$ in $C^s[a,b]$ implies

$$K_s(D^k Lf ; \delta)_{I'} \leq a_{k,L} \cdot K_s\left(f^{(k)} ; \frac{b_{k,s,L}}{a_{k,L}} \cdot \delta\right)_I ,$$

which was our claim. □

In order to formulate the next proposition, we first recall the definition of almost convexity as given by Knoop and Pottinger [18] (see also [11], [29]). For $r \geq 1$ an operator $L: C(I) \to C(I')$ is said to be *almost convex of order* $r-1$, if the following holds:

Let $\mathcal{K}_{I,i} := \{f \in C(I): [x_0, \ldots, x_i ; f] \geq 0$ for any $x_0 < \ldots < x_i \in I\}$, where $[x_0, \ldots, x_i ; f]$ is an i-th order divided difference of f. There exist $p \geq 0$ integers i_j, $1 \leq j \leq p$, satisfying $0 \leq i_1 < \ldots < i_p < r$ such that

$$f \in \left(\bigcap_{j=1}^{p} \mathcal{K}_{I,i_j}\right) \cap \mathcal{K}_{I,r} \quad \text{implies} \quad Lf \in \mathcal{K}_{I',r} .$$

For $p = 0$ we put $\bigcap_{j=1}^{p} \mathcal{K}_{I,i_j} := C(I)$. In this case, $\mathcal{K}_{I,r}$ is mapped by L into $\mathcal{K}_{I',r}$, and L is called *convex of order* $r-1 \geq 0$. For example, convexity of order 0 is just the preservation of monotonicity. Sometimes positive linear operators are said to be *convex of order* -1, a convention to be followed below.

We are now in the state to prove an assertion on the preservation of global smoothness by operators being almost convex of appropriate orders.

Theorem 2.2.
Let $k \geq 0$ and $s \geq 1$ be integers, and let I and I' be given as above. Furthermore, let $L: C^k(I) \to C^k(I')$ be a linear operator having the following properties:

(i) L is almost convex of orders $k-1$ and $k+s-1$,

(ii) L maps $C^{k+s}(I)$ into $C^{k+s}(I')$,

(iii) $L(\Pi_{k-1}) \subseteq \Pi_{k-1}$ and $L(\Pi_{k+s-1}) \subseteq \Pi_{k+s-1}$,

(iv) $L[C^k(I)] \not\subset \Pi_{k-1}$.

Then for all $f \in C^k(I)$ and all $\delta \geq 0$ we have

$$K_s(D^kLf\,;\delta)_{I'} \leq (1/k!) \cdot \|D^kLe_k\| \cdot K_s\left(f^{(k)}\;;\;\frac{1}{(k+s)_s} \cdot \frac{\|D^{k+s}Le_{k+s}\|}{\|D^kLe_k\|} \cdot \delta\right)_I .$$

In the above, e_k is the k-th monomial, $(a)_b$ denotes the *Pochhammer symbol* defined by $(a)_b := \prod_{\beta=0}^{b-1}(a-\beta)$ subject to the usual convention $\prod_{\beta=0}^{-1}(a-\beta) := 1$, and $\Pi_{-1} := \{0\}$.

Proof.
We show that the assumptions of Theorem 2.1 are satisfied. To this end let $\ell \in \{k, k+s\}$ and consider $L: C^\ell(I) \to C^\ell(I')$ which is almost convex of order $\ell - 1$, also satisfying $L(\Pi_{\ell-1}) \subseteq \Pi_{\ell-1}$.

For $\ell = 0$ the operator is positive, maps $C^0(I)$ into $C^0(I')$, and the third assumption is trivially satisfied. For such operators it is known that $\|Lf\|_{I'} \leq \|Le_0\|_{I'} \cdot \|f\|_I$.

For $\ell \geq 1$, define $I_\ell: C(I) \to C^\ell(I)$ by

$$I_\ell(f\,;x) := \int_a^x \frac{(x-t)^{\ell-1}}{(\ell-1)!} \cdot f(t)\,dt .$$

Since L is almost convex of order $\ell - 1$, the operator Q_ℓ given by $Q_\ell := D^\ell \circ L \circ I_\ell$ is linear and positive. The assumption $L[\Pi_{\ell-1}] \subseteq \Pi_{\ell-1}$ implies $Q_\ell D^\ell f = D^\ell Lf$ for all $f \in C^\ell(I)$. Hence

$$\|D^\ell Lf\| = \|Q_\ell D^\ell f\| \leq \|Q_\ell\| \cdot \|D^\ell f\| \quad \text{for all } f \in C^\ell(I).$$

Since Q_ℓ is positive, one has

$$\|Q_\ell\| = \|Q_\ell e_0\| = \left\| \frac{1}{\ell!} \cdot D^\ell \, L \, e_\ell \right\| .$$

Putting now $a_{k,L} := \left\| \frac{1}{k!} D^k \, L \, e_k \right\|$, and

$$b_{k,s,L} := \left\| \frac{1}{(k+s)!} D^{k+s} \, L \, e_{k+s} \right\|$$

yields two nonnegative constants for which the assumptions of Theorem 2.1 are satisfied.

It remains to be shown that $a_{k,L} = \left\| \frac{1}{k!} D^k \, L \, e_k \right\| \neq 0$. Suppose $\left\| \frac{1}{k!} D^k \, L \, e_k \right\| = \|Q_\ell\| = 0$. Then from the above it follows that $D^k Lf = 0$ for all $f \in C^k(I)$, or $L: C^k(I) \rightarrow \Pi_{k-1}$. But this contradicts condition (iv) and hence the proof is complete. □

3. Application to Univariate Bernstein Operators

In this section we apply the general result from Section 2 to the classical Bernstein operators $B_n: C[0,1] \rightarrow \Pi_n$. As a first result we have

Proposition 3.1.
Let $k \geq 0$ and $s \geq 1$ be fixed integers. Then for all $n \geq k+s$, all $f \in C^k[0,1]$ and all $\delta \geq 0$ the following inequality holds:

$$K_s(D^k B_n f \, ; \delta)_{[0,1]} \leq \frac{(n)_k}{n^k} \cdot K_s(f^{(k)} \, ; \frac{(n-k)_s}{n^s} \cdot \delta)_{[0,1]} .$$

Proof.
Since B_n is a polynomial operator, the general assumption and condition (ii) in Theorem 2.2 are satisfied. Due to the representation

$$(D^\ell B_n f)(x) = \frac{(n)_\ell}{n^\ell} \cdot \ell! \sum_{\nu=0}^{n-\ell} \left[\frac{\nu}{n}, \ldots, \frac{\nu+\ell}{n} ; f \right] \cdot \binom{n-\ell}{\nu} x^\nu \cdot (1-x)^{n-\ell-\nu}$$

(see, e.g., Lorentz [20, p. 12]), B_n is (almost) convex of all orders $\ell - 1 \geq -1$, so that (i) is also satisfied.

Since B_n maps a polynomial of degree ℓ to a polynomial of degree $\min\{n,\ell\}$, condition (iii) is satisfied as well, even for all $n \in \mathbb{N}$.

Now consider the k-th monomial $e_k \in C^k[0,1]$. From the assumption that $n \geq k + s$ it follows that $B_n e_k \in \Pi_k \setminus \Pi_{k-1}$, so that (iv) of Theorem 2.2 is also verified.

A representation of the quantities $D^\ell B_n e_\ell$, $\ell \in \{k,k+s\}$, figuring in the inequality of Theorem 2.2 is given by (see, e.g., [11, p. 429])

$$D^\ell B_n e_\ell = \frac{(n)_\ell}{n^\ell} \cdot \ell! \quad .$$

Plugging these expressions into the inequality of Theorem 2.2 yields our claim. □

We now consider two special cases of $s \geq 1$ which are of particular interest. The first is the case $s = 1$ leading to

Proposition 3.2.
Let $k \geq 0$ be a fixed integer. Then for all $n \geq k+1$, $f \in C^k[0,1]$ and $\delta \geq 0$ we have

$$\omega_1(D^k B_n f; \delta) \leq \frac{(n)_k}{n^k} \cdot \tilde{\omega}_1(f^{(k)}; \frac{n-k}{n} \cdot \delta) \leq 1 \cdot \tilde{\omega}_1(f; \delta) .$$

The leftmost inequality is best possible in the sense that for e_{k+1} both sides are equal and do not vanish.

Proof.
Proposition 3.1 gives in this particular case

$$K_1(D^k B_n f ; \delta)_{[0,1]} \leq \frac{(n)_k}{n^k} \cdot K_1(f^{(k)} ; \frac{n-k}{n} \cdot \delta)_{[0,1]} .$$

For the K-functional K_1 it is known from Brudnyĭ's representation theorem (see, e.g., [21, p. 1258]) that

$$K_1(f ; \delta)_{[0,1]} = \frac{1}{2} \cdot \tilde{\omega}_1(f ; 2\delta) .$$

Using this representation on both sides in the inequality involving K_1 leads to the first assertion of Proposition 3.2. Furthermore, for the function $e_{k+1}(x) = x^{k+1}$ it can easily be verified that, for $n \geq k + 1$, both sides of the leftmost inequality in Proposition 3.2 equal $\frac{(n)_{k+1}}{n^{k+1}} \cdot (k+1)! \cdot \delta > 0$ for $\delta > 0$. □

While Proposition 3.2 generalizes an earlier result of Anastassiou and the present authors (see [1]), the following is the extension of Theorem B (and its generalizations) to the simultaneous approximation setting.

Corollary 3.3.

For a fixed integer $k \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(k)} \in \mathrm{Lip}_M(\alpha;[0,1])$ for some $M \geq 0$ and some $0 < \alpha \leq 1$, then $D^k B_n f$ is in the same Lipschitz class.

The second case we will discuss in more detail is $s = 2$. As an immediate consequence of Proposition 3.1 we get

Corollary 3.4.
For a fixed integer $k \geq 0$ one has, for all $n \geq k + 2$, that

$$K_2(D^k B_n f\,;\,\delta\,) \leq \frac{(n)_k}{n^k} \cdot K_2\left(f^{(k)}\,;\,\frac{(n-k)(n-k-1)}{n^2}\cdot\delta\right) \leq K_2(f^{(k)}\,;\,\delta)\,.$$

Proposition 3.2 was derived from Proposition 3.1 by an explicite representation of the K-functional K_1 in terms of the modulus of continuity ω_1. If we wanted to formulate similar statements in terms of higher order moduli of smoothness, we would only be able to use certain equivalence relations between K_s and ω_s for $s > 1$. The problem is that no sharp constants are known in such relations, so that the derived inequalities in terms of ω_s would become more or less irrelevant due to the loss of information. The case $s = 2$ is somewhat exceptional since here we at least know some "reasonable" constants in the equivalence relation. We may use, e.g., that, for $0 \leq \delta \leq \frac{1}{2}$, one has

$$\frac{1}{4}\cdot\omega_2(f;\delta) \leq K_2\left(f\,;\,\frac{1}{4}\cdot\delta^2\right) \leq \frac{9}{8}\cdot\omega_2(f;\delta)\,.$$

The left inequality may be proved using standard techniques, while the right one can be obtained employing the functions $S_\delta(f)$ from Žuk's paper [30] (see Lemma 1 there), also observing the fact that

$$K_2(f;\delta) = K(f;\delta;C[0,1],\,C^2[0,1]) = K(f;\delta;C[0,1],\,W_{2,\infty}[0,1])\,.$$

Here,

$$W_{2,\infty}[0,1] := \left\{f \in C[0,1] : f' \text{ absolutely continuous, } \|f''\|_{L_\infty} < \infty\right\},$$

where

$$\|f''\|_{L_\infty} = \operatorname*{vrai\,sup}_{x\in[0,1]} |f(x)|\,.$$

Similar statements involving $\omega_2(f^{(k)},\delta)$ are obtained if one starts off directly with $\omega_2(D^k B_n f;\delta)$ using Žuk's functions. As the result of this approach we have

Proposition 3.5.
For a fixed integer $k \geq 0$, let $f \in C^k[0,1]$. Then for all $\delta \geq 0$ the Bernstein operators B_n satisfy the inequality

$$\omega_2(D^k B_n f\,;\,\delta) \leq 3 \cdot \frac{(n)_k}{n^k} \cdot \left[1 + \frac{(n-k)(n-k-1)}{2n^2}\right] \cdot \omega_2(f^{(k)};\,\delta\,).$$

In particular, for $k = 0$ we have

$$\omega_2(B_n f\,;\,\delta) \leq 3\left[1 + \frac{n-1}{2n}\right] \cdot \omega_2(f;\delta) \leq 4.5 \cdot \omega_2(f;\delta).$$

Proof.
Let $f \in C^k[0,1]$, $0 < \delta \leq \frac{1}{2}$ be arbitrarily given, and let $|h| \leq \delta$. Then for a typical difference figuring in the definition of $\omega_2(D^k B_n f\,;\,\delta)$ we have

$$|D^k B_n f(x-h) - 2 \cdot D^k B_n f(x) + D^k B_n f(x+h)|$$

$$= |\{D^k B_n(f-g\,;\,x-h) - 2 \cdot D^k B_n(f-g\,;\,x) + D^k B_n(f-g\,;\,x+h)\}$$

$$+ \{D^k B_n(g\,;\,x-h) - 2 \cdot D^k B_n(g\,;\,x) + D^k B_n(g\,;\,x+h)\}|,$$

where $g \in C^k[0,1]$ with $g^{(k)} \in W_{2,\infty}[0,1]$ may be arbitrarily chosen.

The absolute value of the first term in curly parentheses can be estimated from above by

$$4 \cdot \|D^k B_n(f-g)\|_\infty \leq 4 \cdot \frac{(n)_k}{n^k} \cdot \|(f-g)^{(k)}\|_\infty .$$

For the modulus of the second expression in curly brackets we have

$$|D^k B_n(g\,;\,x-h) - 2 \cdot D^k B_n(g\,;\,x) + D^k B_n(g\,;\,x+h)|$$
$$= |D^{k+2} B_n(g\,;\,\xi)| \cdot h^2 \quad (\text{ for some } \xi \text{ between } x-h \text{ and } x+h)$$

$$\leq \|D^{k+2} B_n g\| \cdot h^2$$

$$\leq \frac{(n)_{k+2}}{n^{k+2}} \cdot h^2 \cdot \|g^{(k+2)}\|_{L_\infty}$$

We now substitute the function $g^{(k)} \in W_{2,\infty}[0,1]$ by $S_h(f)$ from Žuk's paper [30], satisfying for $0 < h \leq \frac{1}{2}$ the inequalities

$$\|f - S_h(f)\| \leq \frac{3}{4} \cdot \omega_2(f;h)\,, \quad \text{and} \quad \|S_h(f)''\|_{L_\infty} \leq \frac{3}{2} h^{-2} \cdot \omega_2(f;h)\,.$$

This gives

$$\|(f-g)^{(k)}\| = \|f^{(k)} - g^{(k)}\| \leq \frac{3}{4} \cdot \omega_2(f^{(k)};h)\,,$$

and

$$\|(g)^{(k+2)}\|_{L_\infty} = \|g^{(k)''}\|_{L_\infty} \leq \frac{3}{2} h^{-2} \cdot \omega_2(f^{(k)};h)\,.$$

Combining these estimates leads to

$$\omega_2(D^k B_n f;\delta) \leq 3\,\frac{(n)_k}{n^k} \cdot \left[1 + \frac{1}{2}\,\frac{(n-k)(n-k-1)}{n^2}\right] \cdot \omega_2(f^{(k)};\delta)\,.$$

In particular, for $k = 0$ we have $(n)_k = 1$. Hence in this case we obtain the second assertion of Proposition 3.5. □

If we define Lipschitz classes with respect to the second order modulus by

$$\mathrm{Lip}^*_M(\alpha,[0,1]) := \left\{ f \in C[0,1] : \omega_2(f;\delta) \leq M \cdot \delta^\alpha\,,\ 0 \leq \delta \leq \frac{1}{2} \right\},\quad 0 < \alpha \leq 2\,,$$

then Proposition 3.5 shows that for Bernstein operators one has the inclusion

$$B_n[\mathrm{Lip}^*_M(\alpha,[0,1])] \subseteq \mathrm{Lip}^*_{4.5M}(\alpha,[0,1])\,.$$

As indicated above, the technique used to obtain this subset-relation is most likely not suited to derive "optimal" results like the one of Lindvall. Some numerical evidence shows that it is appropriate to pose the following

Problem 3.6.
Is it true for the classical Bernstein operators B_n that

$$f \in \mathrm{Lip}^*_M(\alpha,[0,1]) \quad \text{implies} \quad B_n f \in \mathrm{Lip}^*_M(\alpha,[0,1])\,,\ 0 < \alpha \leq 2 \ ?$$

In regard to this problem the reader is reminded of the fact that in the Brown/

Elliott/ Paget paper [4] the implication

$$f \in \mathrm{Lip}_M(\alpha;[0,1]) \Rightarrow B_n f \in \mathrm{Lip}_M(\alpha;[0,1]),\ 0 < \alpha \le 1,$$

was proved elementary, i.e., without the detour via a suitable K-functional or a smoothing approach. For the second order modulus of smoothness we were unable to detect an analogous direct approach which might be helpful to get closer to an answer of Problem 3.6.

4. Multivariate Operators

Global smoothness preservation by multivariate approximation operators of various types was already considered in [2] and [5], among others. In this section we shall deal with assertions analogous to those from the previous sections, but in a multivariate setting. We restrict ourselves to giving estimates in terms of first order moduli of continuity.

4.1 Tensor Product Operators

Let $S = \mathop{\times}\limits_{\delta=1}^{d} [a_\delta, b_\delta] \subset \mathbb{R}^d$ denote a generalized cube with non-empty interior.

We equip S with the metric d_1 given by

$$d_1(x,y) = \sum_{\delta=1}^{d} |x_\delta - y_\delta| \quad \text{for } x = (x_\delta) \text{ and } y = (y_\delta).$$

For a multiindex $k = (k_1, \dots, k_d)$ with non-negative integer components k_δ, we denote by $C^{(k_1,\dots,k_d)} = C^{(k_1,\dots,k_d)}(S)$ the space of all real-valued functions f having continuous partial derivatives up to order $(k_1, \dots, k_d)$. The corresponding partial differential operator is usually written as $D^{(k_1,\dots,k_d)}$. For brevity of notation, here and in the following section we also use the following convention: If in $D^{(k_1,\dots,k_d)}$ we have $k_\delta = 0$ for $\delta \ne \sigma$, σ fixed, we will denote the partial differential operator $D^{(0,\dots,0,k_\sigma,0,\dots,0)}$ simply by D^{k_σ}. Furthermore, we put

$$C(S) := C^{(0,\dots,0)}(S),$$

$$C^1(S) := \{f \in C(S) : f \text{ is continuously differentiable with respect to each variable}\}, \text{ and}$$

$$\tilde{C}^{(k_1,\dots,k_d)}(S) := \left\{ g \in C^{(k_1,\dots,k_d)}(S) : D^{(k_1,\dots,k_d)} g \in C^1(S) \right\}.$$

Using the above notation we are ready to formulate the following

Theorem 4.1.
Let S be given as above. Suppose that

$$L : C^{(k_1, \dots, k_d)}(S) \to C^{(k_1, \dots, k_d)}(S)$$

is a linear operator satisfying, for all $f \in C^{(k_1, \dots, k_d)}(S)$,

$$\|D^{(k_1, \dots, k_d)} L f\| \leq a \cdot \|D^{(k_1, \dots, k_d)} f\| \quad , a \neq 0 .$$

Assume, furthermore, that

$$L : \tilde{C}^{(k_1, \dots, k_d)}(S) \to \tilde{C}^{(k_1, \dots, k_d)}(S)$$

so that, for all $g \in \tilde{C}^{(k_1, \dots, k_d)}(S)$, there holds

$$\max_{1 \leq \delta \leq d} \|D^{(k_1, \dots, k_\delta + 1, \dots, k_d)} L g\|$$

$$\leq c \cdot \max_{1 \leq \delta \leq d} \|D^{(k_1, \dots, k_\delta + 1, \dots, k_d)} g\| .$$

Then for all $f \in C^{(k_1, \dots, k_d)}(S)$ and all $t \geq 0$ one has

$$\omega_{d_1}\left(D^{(k_1, \dots, k_d)} L f; t\right) \leq a \cdot \tilde{\omega}_{d_1}\left(D^{(k_1, \dots, k_d)} f ; \frac{c}{a} \cdot t\right).$$

Proof.
Using the definition of ω_{d_1} first observe that

$$\omega_{d_1}\left(D^{(k_1, \dots, k_d)} L f; t\right)$$

$$= \sup\{|D^{(k_1, \dots, k_d)} L f(x) - D^{(k_1, \dots, k_d)} L f(y)| : d_1(x,y) \leq t\}$$

$$\leq 2 \cdot \|D^{(k_1, \dots, k_d)} L f\|$$

$$\leq 2 \cdot a \cdot \|D^{(k_1, \dots, k_d)} f\| .$$

Furthermore, for $g \in \tilde{C}^{(k_1, \dots, k_d)}(S)$,

$$\omega_{d_1}\left(D^{(k_1, \dots, k_d)} L g; t\right)$$

$$= \sup\{|D^{(k_1, \dots, k_d)} L g(x) - D^{(k_1, \dots, k_d)} L g(y)| : d_1(x,y) \leq t\} .$$

For each difference one gets by the mean value theorem

$$|D^{(k_1,\dots,k_d)}Lg(\mathbf{x}) - D^{(k_1,\dots,k_d)}Lg(\mathbf{y})|$$

$$= |(\mathbf{x}-\mathbf{y}) \cdot \nabla D^{(k_1,\dots,k_d)} Lg(\mathbf{y} + \vartheta(\mathbf{x}-\mathbf{y}))| \quad \text{with } 0 < \vartheta < 1$$

$$= \left| \sum_{\delta=1}^{d} (x_\delta - y_\delta) \cdot D^{(\dots, k_\delta+1, \dots)} Lg(\mathbf{y} + \vartheta(\mathbf{x}-\mathbf{y})) \right|$$

$$\leq \sum_{\delta=1}^{d} |x_\delta - y_\delta| \cdot \left| D^{(\dots, k_\delta+1, \dots)} Lg(\mathbf{y} + \vartheta(\mathbf{x}-\mathbf{y})) \right|$$

$$\leq d_1(\mathbf{x},\mathbf{y}) \cdot \max_{1 \leq \delta \leq d} \|D^{(\dots, k_\delta+1, \dots)}Lg\|$$

$$\leq d_1(\mathbf{x},\mathbf{y}) \cdot c \cdot \max_{1 \leq \delta \leq d} \|D^{(k_1,\dots,k_\delta+1,\dots,k_d)}g\| .$$

This implies

$$\omega_{d_1}\left(D^{(k_1,\dots,k_d)}Lg;t\right)$$

$$\leq t \cdot c \cdot \max_{1 \leq \delta \leq d} \|D^{(k_1,\dots,k_\delta+1,\dots,k_d)}g\| .$$

Hence for $f \in C^{(k_1,\dots,k_d)}(S)$ and an arbitrary $g \in \tilde{C}^{(k_1,\dots,k_d)}(S)$ we have

$$\omega_{d_1}\left(D^{(k_1,\dots,k_d)}Lf;t\right)$$

$$\leq \omega_{d_1}\left(D^{(k_1,\dots,k_d)}L(f-g);t\right) + \omega_{d_1}\left(D^{(k_1,\dots,k_d)}Lg;t\right)$$

$$\leq 2 \cdot a \cdot \|D^{(k_1,\dots,k_d)}(f-g)\| + c \cdot t \cdot \max_{1 \leq \delta \leq d} \|D^{(k_1,\dots,k_\delta+1,\dots,k_d)}g\|$$

$$= 2a\left\{\left\|D^{(k_1,\dots,k_d)}(f-g)\right\| + \frac{ct}{2a} \cdot \max_{1 \leq \ell \leq d} \left\|D^{(k_1,\dots,k_\ell+1,\dots,k_d)}g\right\|\right\} .$$

Passing to the infimum gives

$$\omega_{d_1}\left(D^{(k_1,\dots,k_d)}Lf;t\right)$$

$$\leq 2a \cdot K\left(D^{(k_1,\dots,k_d)} f ; \frac{ct}{2a} ; C(S), C^1(S)\right)$$

$$= a \cdot \tilde{\omega}_{d_1}\left(D^{(k_1, \dots, k_d)} f ; \frac{ct}{a}\right) . \qquad \square$$

Of particular interest are tensor product operators defined for functions on S. These can be written as products of parametric extensions of univariate operators. To be more specific, for $1 \le \delta \le d$, let $U_\delta : C^{k_\delta}(I_\delta) \to C^{k_\delta}(I_\delta)$ be linear operators. The parametric extension $U_\delta^{x_\delta} : C^{(k_1,\dots,k_d)} \to C^{(k_1,\dots,k_d)}$ is defined by applying U_δ to a function $f \in C^{(k_1,\dots,k_d)}$ as if all components of f except the δ-th one were fixed, i.e.,

$$U_\delta^{x_\delta} f(x_1, \dots , x_d) := U_\delta f_\delta(x_\delta),$$

where $f_\delta : I_\delta \to \mathbb{R}$, $f_\delta(x_\delta) := f(x_1, \dots , x_d)$ with x_ℓ fixed for $\ell \ne \delta$.
Then the tensor product L of the operators U_δ is given as the product

$$L := U_1^{x_1} \circ U_2^{x_2} \circ \dots \circ U_d^{x_d} .$$

For the proof of Theorem 4.3 below we need the following lemma given in [12, Lemma 3.1] for the case $d = 2$.

Lemma 4.2.
For fixed $1 \le \delta \le d$ and $k_\delta \ge 0$, let $U_\delta : C^{k_\delta}(I_\delta) \to C(I_\delta)$ be a continuous linear operator. Then, for all $\ell \ne \delta$, the partial differential operators D^{κ_ℓ}, $0 \le \kappa_\ell \le k_\ell$, and the parametric extension $U_\delta^{x_\delta}$ commute on $C^{(k_1, \dots ,k_d)}$, i.e.,

$$D^{\kappa_\ell} \circ U_\delta^{x_\delta} = U_\delta^{x_\delta} \circ D^{\kappa_\ell} \quad , 0 \le \kappa_\ell \le k_\ell .$$

Next we show how tensor product operators inherit global smoothness preservation properties from their univariate building blocks.

Theorem 4.3.
Let operators U_δ , $\delta = 1, \dots , d$, and their tensor product L be given as above. Furthermore, suppose that

$$\|D^{k_\delta} U_\delta f\| \le a_\delta \cdot \|f^{(k_\delta)}\|_I , \quad a_\delta \ne 0 , \quad \text{for all } f \in C^{k_\delta}(I_\delta) ,$$

as well as

$$U_\delta : C^{k_\delta+1}(I_\delta) \to C^{k_\delta+1}(I_\delta) \text{ with } \|D^{k_\delta+1}U_\delta\, g\| \le c_\delta \cdot \|g^{(k_\delta+1)}\| .$$

Then for all $f \in C^{(k_1,\dots,k_d)}$ and for all $t \ge 0$ there holds

$$\omega_{d_1}[D^{(k_1,\dots,k_d)} L\, f\, ;\, t] \le \left(\prod_{i=1}^{d} a_i\right) \cdot \tilde{\omega}_{d_1}[D^{(k_1,\dots,k_d)} f\, ;\, m \cdot t] ,$$

where $m := \max\{ c_i/a_i : 1 \le i \le d \}$.

Proof.

We show that for the operator L the assumptions of Theorem 4.1 are satisfied. For an arbitrary $f \in C^{(k_1,\dots,k_d)}$ we obtain, in view of Lemma 4.2, that

$$\|D^{(k_1,\dots,k_d)} L f\|$$

$$= \left\|\left(D^{k_1} \circ \dots \circ D^{k_d} \circ U_1^{x_1} \circ \dots \circ U_d^{x_d}\right) f\right\|$$

$$= \left\|\left(D^{k_1} \circ U_1^{x_1}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) f\right\|$$

$$\le a_1 \cdot \left\|D^{k_1} \circ \left(D^{k_2} \circ U_2^{x_2}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) f\right\|$$

$$= a_1 \cdot \left\|\left(D^{k_2} \circ U_2^{x_2}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right)\left(D^{k_1} f\right)\right\| .$$

Proceeding in an analogous fashion eventually yields

$$\|D^{(k_1,\dots,k_d)} L f\| \le \left(\prod_{i=1}^{d} a_i\right) \cdot \left\|\left(D^{k_d} \circ \dots \circ D^{k_2} \circ D^{k_1}\right) f\right\| = \left(\prod_{i=1}^{d} a_i\right) \cdot \left\|D^{(k_1,\dots,k_d)} f\right\| .$$

For fixed $1 \le \ell \le d$ and all $g \in \tilde{C}^{(k_1,\dots,k_d)}(S)$ we have

$$\|D^{(k_1,\dots,k_\ell+1,\dots,k_d)} L g\|$$

$$= \left\|\left(D^{k_1} \circ \dots \circ D^{k_\ell+1} \circ \dots \circ D^{k_d}\right) \circ \left(U_1^{x_1} \circ \dots \circ U_d^{x_d}\right) g\right\|$$

$$= \left\|\left(D^{k_1} \circ \dots \circ D^{k_{\ell-1}} \circ D^{k_{\ell+1}} \circ \dots \circ D^{k_d}\right) \circ \left(U_1^{x_1} \circ \dots \circ D^{k_\ell+1} \circ U_\ell^{x_\ell} \circ \dots U_d^{x_d}\right) g\right\|$$

$$= \left\|\left(D^{k_1} \circ U_1^{x_1}\right) \circ \dots \circ \left(D^{k_{\ell-1}} \circ U_{\ell-1}^{x_{\ell-1}}\right) \circ \left(D^{k_\ell+1} \circ U_\ell^{x_\ell}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) g\right\|$$

$$\le a_1 \cdot \left\|D^{k_1} \circ \left(D^{k_2} \circ U_2^{x_2}\right) \circ \dots \circ \left(D^{k_\ell+1} \circ U_\ell^{x_\ell}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) g\right\|$$

$$\leq \left(\prod_{i=1}^{\ell-1} a_i\right) \cdot \left\|\left(D^{k_\ell+1} \circ U_\ell^{x_\ell}\right) \circ D^{k_1} \circ \dots \circ D^{k_{\ell-1}} \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) g\right\|$$

$$\leq \left(\prod_{i=1}^{\ell-1} a_i\right) \cdot c_\ell \cdot \left\|\left(D^{k_{\ell+1}} \circ U_{\ell+1}^{x_{\ell+1}}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) \circ D^{k_\ell+1} \circ \left(D^{k_1} \circ \dots D^{k_{\ell-1}}\right) g\right\|$$

$$= \left(\prod_{i=1}^{\ell-1} a_i\right) \cdot c_\ell \cdot \left\|\left(D^{k_{\ell+1}} \circ U_{\ell+1}^{x_{\ell+1}}\right) \circ \dots \circ \left(D^{k_d} \circ U_d^{x_d}\right) \circ \left(D^{k_1} \circ \dots \circ D^{k_{\ell-1}} \circ D^{k_\ell+1}\right) g\right\|$$

$$\leq \left(\prod_{i=1}^{\ell-1} a_i\right) \cdot c_\ell \cdot \left(\prod_{i=\ell+1}^{d} a_i\right) \cdot \left\|D^{(k_1, \dots, k_\ell+1, \dots, k_d)} g\right\|$$

$$= c_\ell \cdot \left(\prod_{i=1, i\neq\ell}^{d} a_i\right) \cdot \left\|D^{(k_1, \dots, k_\ell+1, \dots k_d)} g\right\| .$$

Thus

$$\max_{1\leq\ell\leq d} \left\|D^{(k_1, \dots, k_\ell+1, \dots, k_d)} Lg\right\|$$

$$\leq \max_{1\leq\ell\leq d} \left\{c_\ell \cdot \prod_{i=1, i\neq\ell}^{d} a_i\right\} \cdot \max_{1\leq\ell\leq d} \left\|D^{(k_1, \dots, k_\ell+1, \dots, k_d)} g\right\|$$

Putting $a = \prod_{i=1}^{d} a_i$ and $c = \max_{1\leq\ell\leq d} \left\{c_\ell \cdot \prod_{i=1, i\neq\ell}^{d} a_i\right\}$ in Theorem 4.1 (and thus $\frac{c}{a} = \max\{c_\ell/a_\ell : 1 \leq \ell \leq d\}$) leads to the inequality claimed above. □

As an example we consider tensor product Bernstein operators

$$B_{n_1 \dots n_d} : C([0,1]^d) \to C([0,1]^d) , \quad n_j \in \mathbb{N} \text{ for } 1 \leq j \leq d ,$$

with

$$B_{n_1 \dots n_d} f(x_1, \dots, x_d) = \sum_{i_1=0}^{n_1} \dots \sum_{i_d=0}^{n_d} f\left(\frac{i_1}{n_1}, \dots, \frac{i_d}{n_d}\right) \cdot \prod_{\nu=1}^{d} P_{n_\nu, i_\nu}(x_\nu) ,$$

where $P_{n,i}(x) := \binom{n}{i} x^i (1 - x)^{n-i} , \quad 0 \leq x \leq 1 .$

From the proofs of Theorem 2.2 and Proposition 3.1 it is seen that for any $k \geq 0$ one has for the univariate operators B_n and appropriate functions f and g that

$$\|D^k B_n f\| \le \frac{(n)_k}{n^k} \cdot \|f^{(k)}\| \quad \text{and} \quad \|D^{k+1} B_n g\| \le \frac{(n)_{k+1}}{n^{k+1}} \cdot \|g^{(k+1)}\| .$$

Hence, for the number m from Theorem 4.3 we get $m = \max_{1 \le i \le d} \left\{\frac{n_i - k_i}{n_i}\right\}$ which leads to the statement of

Example 4.4.
Let $f \in C^{(k_1, \dots, k_d)}([0,1]^d)$. Then for the tensor product Bernstein operators we have, for all $t \ge 0$, that

$$\omega_{d_1}(D^{(k_1, \dots, k_d)} B_{n_1, \dots, n_d} f ; t)$$
$$\le \prod_{i=1}^{d} \left(\frac{(n_i)_{k_i}}{n_i^{k_i}}\right) \cdot \tilde{\omega}_{d_1}\left(f^{(k_1, \dots, k_d)} ; \max_{1 \le i \le d} \left\{\frac{n_i - k_i}{n_i}\right\} \cdot t\right) \le \tilde{\omega}_{d_1}(f^{(k_1, \dots, k_d)} ; t) .$$

Note that, for $k_1 = \dots = k_d = 0$, the statement of Example 4.4 reduces to the inequality $\omega_{d_1}(B_{n_1, \dots, n_d} f;t) \le \tilde{\omega}_{d_1}(f;t)$ already shown in [2, Theorem 5]. For $d = 1$ we obtain again the assertion of Proposition 3.2 .

Tensor products of suitable other univariate approximation operators can be dealt with analogously.

4.2 Simplicial Bernstein Operators

In our previous paper [2] we also considered the preservation of global smoothness by Bernstein operators over simplices. We recall their definition.

Let $\Delta_d := \left\{(x_1, \dots, x_d) \in [0,1]^d : \sum_{i=1}^{d} x_i \le 1\right\}$ denote the standard simplex in $\mathbb{R}^d$.

For $f \in C(\Delta_d)$, $n \in \mathbb{N}$, the simplicial Bernstein operators (see, e.g., [8], [20, p. 51], [25], [26]) are given by

$$B_n^{\Delta} f(x_1, \dots, x_d) := \sum_{\substack{j_1 + \dots + j_d \le n \\ j_i \in \mathbb{N}_0}} f\left(\frac{j_1}{n}, \dots, \frac{j_d}{n}\right) \cdot P_{n,j}(x_1, \dots, x_d) ,$$

where $j = (j_1, \dots, j_d)$ is a multi-index and

$$P_{n,j}(x_1, \dots, x_d) = \frac{n!}{j_1! \dots j_d! (n - j_1 - \dots - j_d)!} \cdot \prod_{i=1}^{d} x_i^{j_i} \cdot \left[1 - \sum_{i=1}^{d} x_i\right]^{n - j_1 - \dots - j_d}$$

For the mixed partial derivatives of $B_n^{\Delta} f$ one has (see [25])

$$D^{(k_1, \ldots, k_d)} B_n^{\Delta} f(x_1, \ldots, x_d)$$

$$= (n)_{k_1+\ldots+k_d} \sum_{\substack{j_1+\ldots+j_d \\ \leq n-k_1-\ldots-k_d}} \Delta_{1/n}^{k_1,\ldots,k_d} f\left(\frac{j_1}{n}, \ldots \frac{j_d}{n}\right) \cdot P_{n-k_1-\ldots-k_d, j}(x_1, \ldots, x_d) .$$

Here,

$$\Delta_{1/n}^{k_1,\ldots,k_d} = \prod_{i=1}^{d} \Delta_{1/n}^{k_i} ,$$

where $\Delta_{1/n}^{r_i}$ is an r_i-th backward difference of f with step size $\frac{1}{n}$ taken with respect to the variable x_i .

From the mean value theorem we get for any sufficiently often differentiable f and n large enough that

$$\left\|\Delta_{1/n}^{k_1,\ldots,k_d} f\right\| \leq \frac{1}{n^{k_1+\ldots+k_d}} \left\|D^{(k_1,\ldots,k_d)} f\right\| .$$

Thus

$$\left\|D^{(k_1,\ldots,k_d)} B_n^{\Delta} f\right\| \leq \frac{(n)_{k_1+\ldots+k_d}}{n^{k_1+\ldots+k_d}} \left\|D^{(k_1,\ldots,k_d)} f\right\| .$$

Hence, for g being sufficiently often differentiable, we have

$$\omega_{d_1}\left(D^{(k_1,\ldots,k_d)} B_n^{\Delta} f \,;\, t\right) \leq \omega_{d_1}\left(D^{(k_1,\ldots,k_d)} B_n^{\Delta} (f-g) \,;\, t\right) + \omega_{d_1}\left(D^{(k_1,\ldots,k_d)} B_n^{\Delta} g \,;\, t\right)$$

$$\leq 2 \left\|D^{(k_1,\ldots,k_d)} B_n^{\Delta} (f-g)\right\| + t \cdot \max_{1\leq i\leq d} \left\|D^{(k_1,\ldots k_{i-1}, k_i+1, k_{i+1}, \ldots, k_d)} B_n^{\Delta} g\right\|$$

$$\leq 2 \frac{(n)_{k_1+\ldots+k_d}}{n^{k_1+\ldots+k_d}} \left\|D^{(k_1,\ldots,k_d)} (f-g)\right\| + t \frac{(n)_{k_1+\ldots+k_d+1}}{n^{k_1+\ldots+k_d+1}} \left\|D^{(k_1,\ldots, k_{i-1}, k_i+1, k_{i+1},\ldots, k_d)} g\right\| .$$

Passing to the infimum over all functions $g \in \tilde{C}^{(k_1, \ldots, k_d)}(S)$ gives

$$\omega_{d_1}\left(D^{(k_1,\ldots,k_d)} B_n^{\Delta} f \,;\, t\right) \leq 2 \frac{(n)_{k_1+\ldots+k_d}}{n^{k_1+\ldots+k_d}} K\left(D^{(k_1,\ldots,k_d)} f \,;\, \frac{n-k_1-\ldots-k_d}{n} \,;\, C(S) \,,\, C^1(S)\right) .$$

Here $C(S)$ and $C^1(S)$ are defined as in Section 4.1, but for $S = \Delta_d$.

An application of Lemma 6 in [2] in combination with the representation theorem of Brudnyĭ leads to

Theorem 4.5.

For the simplicial Bernstein operators B_n^{Δ} we have for all $f \in C^{(k_1,\dots,k_d)}(\Delta_d)$, all sufficiently large n, and all $t \geq 0$ that

$$\omega_{d_1}\left(D^{(k_1,\dots,k_d)} B_n^{\Delta} f \,;\, t\right)$$

$$\leq \frac{(n)_{k_1+\dots+k_d}}{n^{k_1+\dots+k_d}} \cdot \tilde{\omega}_{d_1}\left(D^{(k_1,\dots,k_d)} f \,;\, \frac{n-k_1-\dots-k_d}{n} \cdot t\right) \leq \tilde{\omega}_{d_1}\left(D^{(k_1,\dots,k_d)} f \,;\, t\right) .$$

Note that, for $k_1 = \dots = k_d = 0$, the statement of Theorem 4.5 becomes that of Theorem 3(i) in [2]. Also notice that for $d = 1$ (univariate Bernstein operators defined on $C^k[0,1]$) the above inequality turns into that of Proposition 3.2 in this note.

Remark 4.6.
It is also possible to prove assertions along the lines of this section for Boolean sum operators; see our note [5] in order to get an idea of what can be done in this direction. Natural tools to formulate such estimates are higher order mixed moduli of continuity as used in [13] and the higher order mixed K-functionals from [6]. However, inequalites for these kinds of approximation operators involving derivatives become rather lengthy so that we refrain from presenting them here.

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